The Electronic Journal of Mathematics and Technology, Volume 15, Number 1, ISSN 1933-2823

# Some Elementary Results Related to the Cauchy's Mean Value Theorem

*German Lozada-Cruz*
e-mail: german.lozada@unesp.br
Departamento de Matemática
Instituto de Biociências, Letras e Ciências Exatas (IBILCE)
Universidade Estadual Paulista (UNESP)
15054-000 São José do Rio Preto, São Paulo, Brazil

**Abstract**

*In this note we prove some elementary results of Cauchy's mean value theorem. The main tools employed to get these are auxiliary functions.*

## 1 Introduction

We know that mean value theorems are important tools in real analysis. The first one that we learn is the famous Lagrange's mean value theorem ([2, Theorem 2.3] or [10, Theorem 4.12] e.g.) and it asserts that a function $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on $(a, b)$, then there exists $\eta \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(\eta). \tag{1}$$

This mean value theorem is used to solve a great variety of problems in optimization, economics, etc.

If $f(a) = f(b)$ in (1), then the Lagrange's mean value theorem reduces to Rolle's theorem ([10, Theorem 4.11]). The equivalence between Rolle's and Lagrange's mean value theorems has been proved for example in [11, Theorem B].

In 1958, T. M. Flett [1] proved a variant of Lagrange's mean value theorem. Other authors obtained variants of Lagrange's mean value theorem (see [14] and [4] for example).

In 1977, R. E. Myers [9] proved that there are nine possible quotients in (1) having one of the values $f(b) - f(a)$, $f(\eta) - f(a)$, $f(b) - f(\eta)$ for numerators, and one of $b - a$, $\eta - a$, $b - \eta$ for denominators.

The second mean value theorem is the Cauchy's mean value theorem ([10, Theorem 4.14], [12, Theorem 2.17]), which is a generalization of the Lagrange's mean value theorem. It establishes the relationship between the derivatives of two functions and the variation of these functions on a finite interval.

**Theorem 1 (Cauchy's Mean Value Theorem)** *Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions on $[a, b]$, differentiable on $(a, b)$ and $g'(x) \neq 0$ for all $x \in (a, b)$. Then, there exists $\eta \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\eta)}{g'(\eta)}. \tag{2}$$

For a geometric interpretation of Cauchy's mean value theorem consider the curve $\gamma : [a, b] \to \mathbb{R}^2$ given by $\gamma(t) = (f(t), g(t))$. According to the Cauchy's mean value theorem, there is a point $C = (f(\eta), g(\eta))$ on the curve $\gamma$ where the tangent is parallel to the chord joining the points $A = (f(a), g(a))$ and $B = (f(b), g(b))$ of the curve (see Figure 1).

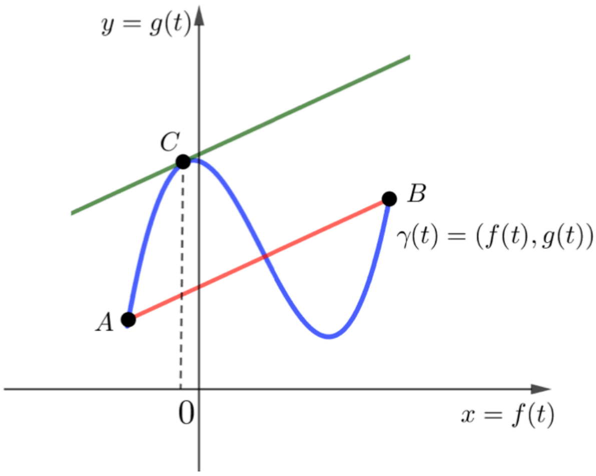


Figure 1: Geometric interpretation of Cauchy's mean value theorem

W.-C. Yang used technological tools and gave a geometric interpretation of Cauchy's mean value theorem (see [15]).

In 2000, E. Wachnicki (see [16, Theorem 1.3]) proved the following variant of Cauchy's mean value theorem.

**Theorem 2 (Wachnicki's Theorem)** *Let $f, g : [a, b] \to \mathbb{R}$ be differentiable functions on $[a, b]$. Suppose that $g'(x) \neq 0$ for all $x \in [a, b]$ and*

$$\frac{f'(a)}{g'(a)} = \frac{f'(b)}{g'(b)}. \tag{3}$$

*Then, there exists $\eta \in (a, b)$ such that*

$$\frac{f(\eta) - f(a)}{g(\eta) - g(a)} = \frac{f'(\eta)}{g'(\eta)}. \tag{4}$$

**Remark 3** *If g(x) = x, then Wachnicki's theorem reduces to Flett's theorem.*

For applications of Wachnicki's Theorem in the study of some Volterra operators type see [7]. For another applications of mean value theorems involving linear integral operators see [6].

In 2014, C. Tan and S. Lin [13] obtained the Cauchy's mean value theorem and Wachnicki's theorem under the condition $\int_a^b f(x)g'(x)dx = \frac{(f(b)+f(a))(g(b)-g(a))}{2}$.

In [3] we proved the following variant of Wachnicki's Theorem.

**Theorem 4 ([3, Theorem 2.4])** *Let $f, g : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$. Suppose that $g'(x) \neq 0$ for all $x \in [a,b]$ and*

$$\frac{f'(a)}{g'(a)} = \frac{f'(b)}{g'(b)}. \tag{5}$$

*Then, there exists $\eta \in (a,b)$ such that*

$$\frac{f(b)-f(\eta)}{g(b)-g(\eta)} = \frac{f'(\eta)}{g'(\eta)}. \tag{6}$$

**Proof.** The proof follows the same reasoning of Wachnicki's Theorem using the function $G : [a,b] \to \mathbb{R}$ given by

$$G(x) = \begin{cases} \frac{f(b)-f(x)}{g(b)-g(x)}, & \text{if } x \in [a,b) \\ \\ \frac{f'(b)}{g'(b)}, & \text{if } x = b. \end{cases}$$

□

**Remark 5** *If $g(x) = x$, then Theorem 4 reduces to Meyer's theorem*

For some applications of Theorem 4 involving Volterra operators type see [5].

There are nine possible quotients having one of the values $f(b) - f(a)$, $f(\eta) - f(a)$, $f(b) - f(\eta)$ for numerators, and one of $g(b) - g(a)$, $g(\eta) - g(a)$, $g(b) - g(\eta)$ for denominators. The theorems mentioned above show that, under appropriate conditions, three of these nine quotients are equal to $\frac{f'(\eta)}{g'(\eta)}$.

In this note we consider the other six cases, i.e., we give conditions which guarantee the existence of a number $\eta \in (a,b)$ for which $\frac{f'(\eta)}{g'(\eta)}$ equals the desired quotient (see Section 2).

# 2 Elementary results of Cauchy's Mean Value Theorem

In this section, we prove elementary results of Cauchy's Mean Value Theorem. Also, we give some simple examples as applications of our results.

**Theorem 6** *Let $f, g : [a,b] \to \mathbb{R}$ be continuous functions on $[a,b]$, differentiable on $(a,b)$ and $g'(x) \neq 0$ for all $x \in (a,b)$, then there exists $\eta \in (a,b)$ such that*

$$\frac{f(b)-f(\eta)}{g(\eta)-g(a)} = \frac{f'(\eta)}{g'(\eta)}. \tag{7}$$

**Proof.** Consider the function $h : [a, b] \to \mathbb{R}$ given by

$$h(x) = f(b)g(x) - (g(x) - g(a))f(x).$$

The function $h$ is continuous on $[a, b]$, differentiable on $(a, b)$ and

$$h'(x) = (f(b) - f(x))g'(x) - (g(x) - g(a))f'(x).$$

Then by Lagrange's mean value theorem there is $\eta \in (a, b)$ such that

$$h(b) - h(a) = h'(\eta)(b - a), \tag{8}$$

which implies that

$$\begin{aligned} f(b)g(b) - (g(b) - g(a))f(b) - f(b)g(a) &= \Big\{(f(b) - f(\eta)g'(\eta) - (g(\eta) - g(a))f'(\eta)\Big\}(b - a) \\ 0 &= \Big\{(f(b) - f(\eta))\, g'(\eta) - (g(\eta) - g(a))f'(\eta)\Big\}(b - a). \end{aligned}$$

Since $b - a > 0$, we get (7). □

**Remark 7** *Using the auxiliary function* $\Psi(t) = \big[g(t) - g(a)\big]\,\big[f(b) - f(t)\big]$, $t \in [a, b]$ *we can also prove* Theorem 6.

**Theorem 8** *Let* $f, g : [a, b] \to \mathbb{R}$ *be continuous functions on* $[a, b]$, *differentiable on* $(a, b)$ *and* $g'(x) \neq 0$ *for all* $x \in (a, b)$, *then there exists* $\eta \in (a, b)$ *such that*

$$\frac{f(\eta) - f(a)}{g(b) - g(\eta)} = \frac{f'(\eta)}{g'(\eta)}. \tag{9}$$

**Proof.** Consider the function $h : [a, b] \to \mathbb{R}$ given by

$$h(x) = f(a)g(x) + \big[g(b) - g(x)\big]f(x)$$

and repeat the same reasoning of the proof of Theorem 6. □

**Remark 9** *Using the auxiliary function* $\varphi(t) = \big[g(b) - g(t)\big]\,\big[f(t) - f(a)\big]$, $t \in [a, b]$, J. Matkowski and I. Pawlikowska *proved* Theorem 8 (*see* $[8, Theorem\ 1]$ ).

In proof of the following theorems we use the Intermediate the Value Theorem and for ease of reference we state it.

**Theorem 10 (Intermediate Value Theorem)** *Let* $h : [a, b] \to \mathbb{R}$ *be a continuous function. If* $d$ *is any number strictly between* $h(a)$ *and* $h(b)$, *then there exists* $\eta \in (a, b)$ *such that* $h(\eta) = d$.

**Definition 11** *If* $L_1$ *and* $L_2$ *are lines with slopes* $m_1$ *and* $m_2$ *respectively, we say that* $L_1$ *is steeper that* $L_2$ *if* $|m_1| > |m_2|$.

**Theorem 12** *Let $f, g : [a, b] \to \mathbb{R}$ be differentiable functions on $[a, b]$. Suppose that $g'(x) \neq 0$ for all $x \in [a, b]$ and*

$$\Big[\big(f(b) - f(a)\big)g'(a)\Big]\Big[\big(f(b) - f(a)\big)g'(b) - \big(g(b) - g(a)\big)f'(b)\Big] < 0. \tag{10}$$

*Then there exists $\eta \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{g(\eta) - g(a)} = \frac{f'(\eta)}{g'(\eta)}. \tag{11}$$

**Proof.** Consider the function $h : [a, b] \to \mathbb{R}$ given by

$$h(x) = \big(f(b) - f(a)\big)\, g'(x) - \big(g(x) - g(a)\big) f'(x).$$

The function $h$ is continuous on $[a, b]$ and from (10) we conclude that $h(a)$ and $h(b)$ have opposite signs. Then by the Intermediate Value Theorem there exists $\eta \in (a, b)$ such that $h(\eta) = 0$, which implies

$$h(\eta) = 0 \Leftrightarrow (f(b) - f(a))g'(\eta) - (g(\eta) - g(a))f'(\eta) = 0 \Leftrightarrow \frac{f(b) - f(a)}{g(\eta) - g(a)} = \frac{f'(\eta)}{g'(\eta)}.$$

□

The condition expressed by the inequality (10) in Theorem 12 has a geometric interpretation: the tangent line at $(f(b), g(b))$ to the curve $\gamma(t) = (f(t), g(t))$, $t \in [a, b]$, must be steeper than the secant line from $(f(a), g(a))$ to $(f(b), g(b))$. Furthermore the two lines have slopes of the same sign.

**Example 13** *Consider the functions $f, g : [0, 3] \to \mathbb{R}$ given by $f(x) = x^2 + 4x - 4$ and $g(x) = -x + 1$.*

*It is easy to see that $f$ and $g$ are differentiable functions on $[0, 3]$, $f'(x) = 2x + 4$ and $g'(x) = -1 \neq 0$, $\forall x \in [0, 3]$. Also*

$$\begin{aligned}&\Big[\big(f(3) - f(0)\big)g'(0)\Big]\Big[\big(f(3) - f(0)\big)g'(3) - \big(g(3) - g(0)\big)f'(3)\Big] \\ &= \big[21(-1)\big]\big[21(-1) - (-2 - 1)10\big] = -21\big[-21 + 30\big] = -21(9) < 0.\end{aligned}$$

*Then, from* Theorem 12 *there exists $\eta \in (0, 3)$ such that*

$$\frac{f(3) - f(0)}{g(\eta) - g(0)} = \frac{f'(\eta)}{g'(\eta)},$$

*which implies that*

$$\begin{aligned}\frac{21}{-\eta} = \frac{2\eta + 4}{-1} &\Leftrightarrow 21 = \eta(2\eta + 4) \\ &\Leftrightarrow 2\eta^2 + 4\eta - 21 = 0. \end{aligned} \tag{12}$$

*Thus, $\eta = -1 + \frac{\sqrt{46}}{2} \approx 2.39 \in (0, 3)$.*

We put here the *octave* code to find $\eta$ in (12) using bisection method where use a function $f(\eta) = 2\eta^2 + 4\eta - 21$.

```
=============================================================
 clear all
tol = 1.e-6;
a = 0.0; b = 3.0;
nmax = 18;
itcount = 0; error = 1.0;
xval = linspace(a,b,20);
fval = 2*xval.^2 +4*xval -21;
plot(xval,fval,'b');
grid on; hold on;
func=@(x)2*x^2+4*x-21;
while (itcount <=nmax && error >= tol)
itcount = itcount + 1;
x = a + (b-a)/2;
z(itcount) = x;
iter(itcount) = itcount;
fa = func(a);
fb = func(b);
fx = func(x);
error = abs(fx);
if (error <tol)
x_final = x;
else
if (fa*fx < 0)
b = x;
else
a = x;
end
end
plot(z(1:itcount),zeros(itcount,1),'r+');
end
if (itcount < nmax);
val = func(x);
fprintf(1,'Converged solution after
fprintf(1,' is
else fprintf(1,'Not converged after
end
[iter' z']
=============================================================
```

Example 13 is a simple example where we show that condition (10) holds. See Figure 2 for an illustration of this example.

The proofs of the remaining three theorems are similar to the proof of Theorem 12. We shall give

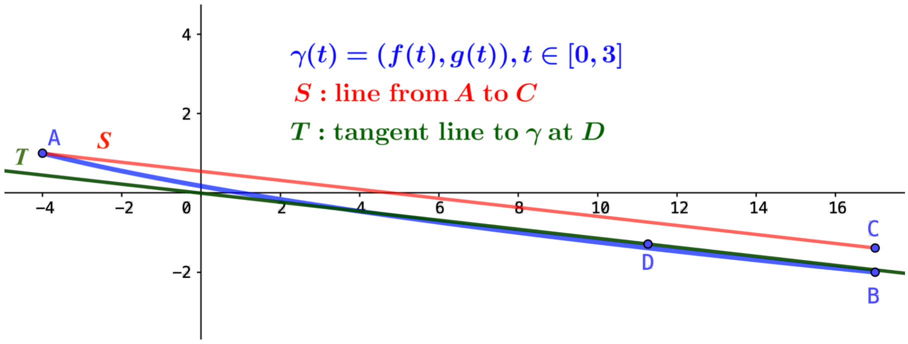


Figure 2: Curve $\gamma(t) = (f(t), g(t), t \in [0,3]$, $\eta \approx 2.39$, $A = (f(0), g(0)) = (-4, 1)$, $B = (f(3), g(3)) = (17, -2)$, $C = (f(3), g(\eta)) = (17, -1.39)$, $D = (f(\eta), g(\eta))$ and $T$ parallel to $S$.

only the function $h$ that is used in the proofs.

**Theorem 14** *Let $f, g : [a, b] \to \mathbb{R}$ be differentiable functions on $[a, b]$. Suppose that $g'(x) \neq 0$ for all $x \in [a, b]$ and*

$$\Big[(f(b) - f(a))g'(b)\Big]\Big[(f(b) - f(a))g'(a) - (g(b) - g(a))f'(a)\Big] < 0. \tag{13}$$

*Then there exists $\eta \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(\eta)} = \frac{f'(\eta)}{g'(\eta)}. \tag{14}$$

**Proof.** Let $h(x) = \big(f(b) - f(a)\big)\, g'(x) - \big(g(b) - g(x)\big)\, f'(x)$. □

The condition expressed by the inequality (13) in Theorem 14 has a geometric interpretation similar to that of Theorem 12.

**Example 15** *Consider the functions $f, g : [0, 3] \to \mathbb{R}$ given by $f(x) = -x+1$ and $g(x) = x^2+4x-4$.*

*It is easy to see that $f$ and $g$ are differentiable functions on $[0, 3]$, $f'(x) = -1$ and $g'(x) = 2x + 4 \neq 0$, $\forall x \in [0, 3]$. Also*

$$\begin{aligned}&\Big[\big(f(3) - f(0)\big)g'(3)\Big]\Big[\big(f(3) - f(0)\big)g'(0) - \big(g(3) - g(0)\big)f'(0)\Big] \\ &= \big[-3(10)\big]\big[-3(4) - 21(-1)\big] = -30(9) < 0.\end{aligned}$$

*Then, from* Theorem 14 *there exists $\eta \in (0, 3)$ such that*

$$\frac{f(3) - f(0)}{g(3) - g(\eta)} = \frac{f'(\eta)}{g'(\eta)},$$

*which implies that*

$$\frac{-3}{17 - (\eta^2 + 4\eta - 4)} = \frac{-1}{2\eta + 4} \Leftrightarrow 3(2\eta + 4) = 21 - \eta^2 - 4\eta \Leftrightarrow \eta^2 + 10\eta - 9 = 0.$$

*Thus, $\eta = -5 + \sqrt{34} \approx 0.83 \in (0, 3)$.*

Example 15 is a simple example where we show that condition (13) holds.

**Theorem 16** *Let $f, g : [a, b] \to \mathbb{R}$ be differentiable functions on $[a, b]$. Suppose that $g'(x) \neq 0$ for all $x \in [a, b]$ and*

$$\Big[(g(b) - g(a))f'(a)\Big]\Big[(f(b) - f(a))g'(b) - (g(b) - g(a))f'(b)\Big] > 0. \tag{15}$$

*Then there exists $\eta \in (a, b)$ such that*

$$\frac{f(\eta) - f(a)}{g(b) - g(a)} = \frac{f'(\eta)}{g'(\eta)}. \tag{16}$$

**Proof.** Let $h(x) = \big(f(x) - f(a)\big)\, g'(x) - \big(g(b) - g(a)\big) f'(x)$. □

**Example 17** *Consider the functions $f, g : [0, 3] \to \mathbb{R}$ given by $f(x) = x + 1$ and $g(x) = x^2 + 4x - 4$.*

*It is easy to see that $f$ and $g$ are differentiable functions on $[0, 3]$, $f'(x) = 1$ and $g'(x) = 2x + 4 \neq 0$, $\forall x \in [0, 3]$. Also*

$$\begin{aligned}&\Big[\big(g(3) - g(0)\big)f'(3)\Big]\Big[\big(f(3) - f(0)\big)g'(3) - \big(g(3) - g(0)\big)f'(3)\Big]\\ &= \big[21(1)\big]\big[3(10) - 21(1)\big] = 21(9) > 0.\end{aligned}$$

*Then, from* Theorem 14 *there exists $\eta \in (0, 3)$ such that*

$$\frac{f(\eta) - f(0)}{g(3) - g(0)} = \frac{f'(\eta)}{g'(\eta)},$$

*which implies that*

$$\frac{\eta}{21} = \frac{1}{2\eta + 4} \Leftrightarrow \eta(2\eta + 4) = 21 \Leftrightarrow 2\eta^2 + 4\eta - 21 = 0.$$

*Thus, $\eta = -1 + \sqrt{\frac{23}{2}} \approx 2.39 \in (0, 3)$.*

Example 17 is a simple example where we show that condition (15) holds.

Note that in Example 13 and in Example 17 we have the same $\eta$, this is because the fractions $\frac{21}{-\eta} = \frac{2\eta+4}{-1}$ and $\frac{\eta}{21} = \frac{1}{2\eta+4}$ are equivalent. More generally in Example 13 we can take $f(x) = x^2 + 4x + c$ and $g(x) = -x + c'$, where $c, c' \in \mathbb{R}$, then we have the same $\eta \approx 2.39$, and in Example 17 we can take $f(x) = x + c_1$ and $g(x) = x^2 + 4x + c_2$, where $c_1, c_2 \in \mathbb{R}$, then we have the same $\eta \approx 2.39$.

**Theorem 18** *Let $f, g : [a, b] \to \mathbb{R}$ be differentiable functions on $[a, b]$. Suppose that $g'(x) \neq 0$ for all $x \in [a, b]$ and*

$$\Big[\big(g(b) - g(a)\big)f'(b)\Big]\Big[\big(f(b) - f(a)\big)g'(a) - \big(g(b) - g(a)\big)f'(a)\Big] > 0. \tag{17}$$

*Then there exists $\eta \in (a, b)$ such that*

$$\frac{f(b) - f(\eta)}{g(b) - g(a)} = \frac{f'(\eta)}{g'(\eta)}. \tag{18}$$

**Proof.** Let $h(x) = \big(f(b) - f(x)\big)\, g'(x) - \big(g(b) - g(a)\big) f'(x)$. $\square$

**Example 19** *Consider the functions $f, g : [0,3] \to \mathbb{R}$ given by $f(x) = x^2 + 3x$ and $g(x) = x + 1$.*
*It is easy to see that $f$ and $g$ are differentiable functions on $[0,3]$, $f'(x) = 2x + 3$ and $g'(x) = 1 \neq 0$, $\forall x \in [0,3]$. Also*

$$\Big[\big(g(3) - g(0)\big) f'(3)\Big] \Big[\big(f(3) - f(0)\big) g'(0) - \big(g(3) - g(0)\big) f'(0)\Big]$$
$$= [3(9)]\,[18(1) - 3(3)] = 27(9) > 0.$$

*Then, from* Theorem 18 *there exists $\eta \in (0,3)$ such that*

$$\frac{f(3) - f(\eta)}{g(3) - g(0)} = \frac{f'(\eta)}{g'(\eta)},$$

*which implies that*

$$\frac{18 - \eta^2 - 3\eta}{3} = \frac{2\eta + 3}{1} \Leftrightarrow 18 - \eta^2 - 3\eta = 3(2\eta + 3) \Leftrightarrow \eta^2 + 9\eta - 9 = 0.$$

*Thus, $\eta = \frac{-9 + \sqrt{117}}{2} \approx 0.91 \in (0,3)$.*

Example 19 is a simple example where we show that condition (17) holds.

# 3 Conclusion

In this note we proved some variants of Cauchy's mean value theorem that are less known and are not studied in a Calculus course or in a first Real Analysis course. Mainly in this note we have shown six variants of Cauchy's mean value theorem (Theorem 6, Theorem 8, Theorem 12, Theorem 14, Theorem 16 and Theorem 18).

Let $f, g : [a,b] \to \mathbb{R}$ be functions. Assuming the continuity of $f$ and $g$ on $[a,b]$, differentiability on $(a,b)$ and $g'(x) \neq 0$ for all $x \in (a,b)$ then we get the Theorem 6 and Theorem 8.

Assuming the differentiability of $f$ and $g$ on $[a,b]$ and $g'(x) \neq 0$ for all $x \in [a,b]$ and using the definition of $L_1$ is steeper that $L_2$ if $|m_1| > |m_2|$, where $L_1$ and $L_2$ are lines with slopes $m_1$ and $m_2$ respectively, we get Theorem 12, Theorem 14, Theorem 16 and Theorem 18.

These results can be seen as elementary but we can't find these in the Analysis textbooks or in a journal of mathematical analysis.


# Acknowledgments

The author sincerely thanks the anonymous reviewers for their valuable suggestions improving this note.

Also, we would like to thank to Prof. C. E. Rubio-Mercedes (UEMS, Dourados-MS, Brazil) for help me with *octave* online code (`https://octave-online.net/`).

# Funding

This work was partially supported by FAPESP grant: 15/24095-6.

# References

[1] Flett, T.M. *A mean value problem*, Math. Gazette, 42 (1958), 38-39.

[2] Lang, S., *Undergraduate Analysis*. Second edition. Undergraduate Texts in Mathematics. Springer-Verlag, New York, 2005.

[3] Lozada-Cruz, G., *Some variants of Cauchy's mean value theorem*. Int. J. Math. Ed. Sci. Tech. Published online: 23 Dec 2019.

[4] Lozada-Cruz, G., *Some variants of Lagrange's mean value theorem*, Sel. Mat. 7 (1): 144-150, 2020.

[5] Lozada-Cruz, G., *Some applications of Cauchy's mean value theorem*, 2020. Submitted for publication.

[6] Lupu, C.; Lupu, T., *Mean value theorems for some linear integral operators*, Electron. J. Diff. Eqn., 2009, no. 117, pp. 1-15.

[7] Lupu, C., *Mean value problems of Flett type for a Volterra operator*, Electron. J. Diff. Eqn., 2013, No. 53, pp. 1–7.

[8] Matkowski, J.; Pawlikowska, I., *Homogeneous means generated by a mean-value theorem*, Journal Mathematical Inequalities, Vol. 4, No. 4 (2010), 467–479.

[9] Myers, R.E. *Some elementary results related to the mean value theorem*, The Two-Year College Mathematics Journal, Vol. 8, No. 1 (1977), pp. 51-53.

[10] Protter, M.H.; Morrey Jr., C.B., *A first course in real analysis*. Second edition. Undergraduate Texts in Mathematics, Springer-Verlag New York, Inc., 1991.

[11] Qazi, M. A., *The mean value theorem and analytic functions of a complex variable*, J. Math. Anal. Appl. 324 (2006), 30–38.

[12] Sahoo, P.K.; Riedel, T., *Mean Value Theorems and Functional Equations*, World Scientific, River Edge, NJ. 1998.

[13] Tan, C.; Li, S., *Some new mean value theorems of Flett type*, Int. J. Math. Ed. Sci. Tech., 45:7 (2014), 1103-1107.

[14] Trahan, D.H., *A new type of mean value theorem*. Math. Mag. 39 (1966), pp. 264–268.

[15] Yang, W.-C., *Revisit Mean Value, Cauchy Mean Value and Lagrange Remainder Theorems*, The Electronic Journal of Mathematics and Technology, Volume 1, Issue 2, 2007.

[16] Wachnicki, E., *Une variante du thï¿½orï¿½me de Cauchy de la valeur moyenne*, Demonstratio Math. 33.4 (2000), 737–740.